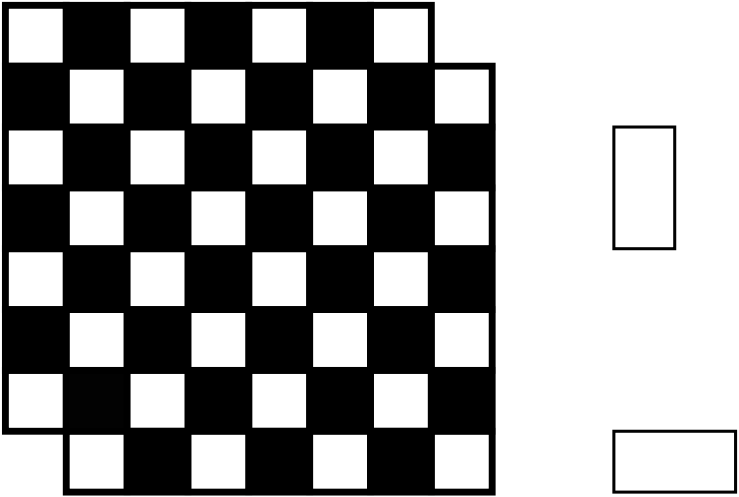
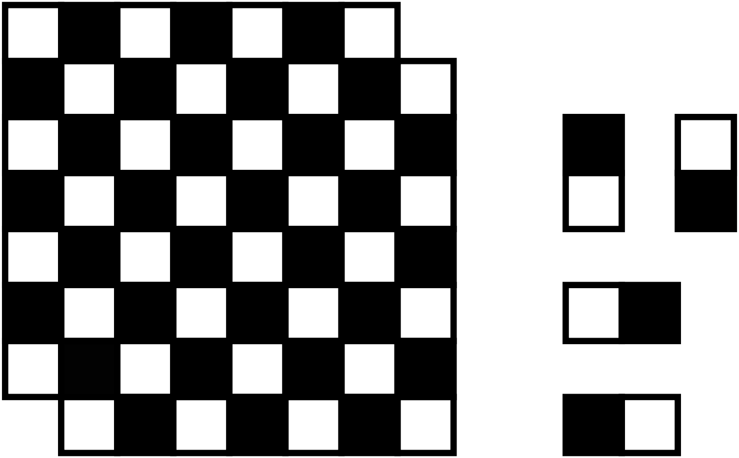
A tiling problem

- Q: Can this board be tiled by dominoes?



A tiling problem

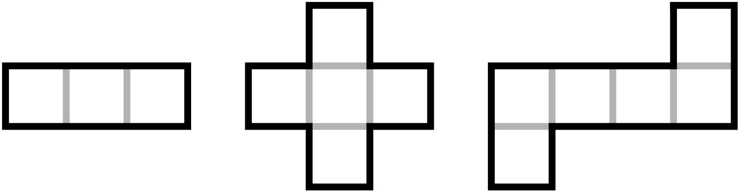
- Q: Can this board be tiled by dominoes?



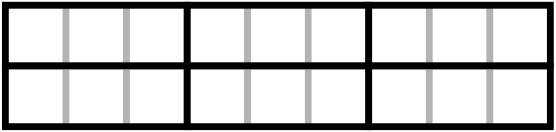
- A: No, each domino must cover 1 black square and 1 white square, and there are 30 black squares and 32 white squares.

Another tiling problem

- Q: For which m, n can an $m \times n$ rectangle be tiled with copies of these tiles (rotations and reflections allowed)?

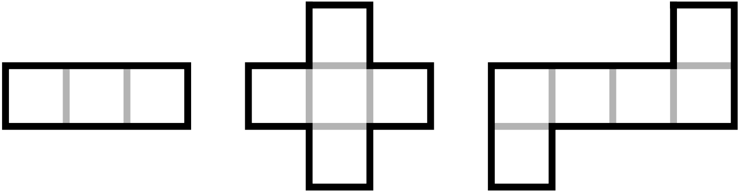


- If mn is a multiple of 3, then the rectangle can be tiled.



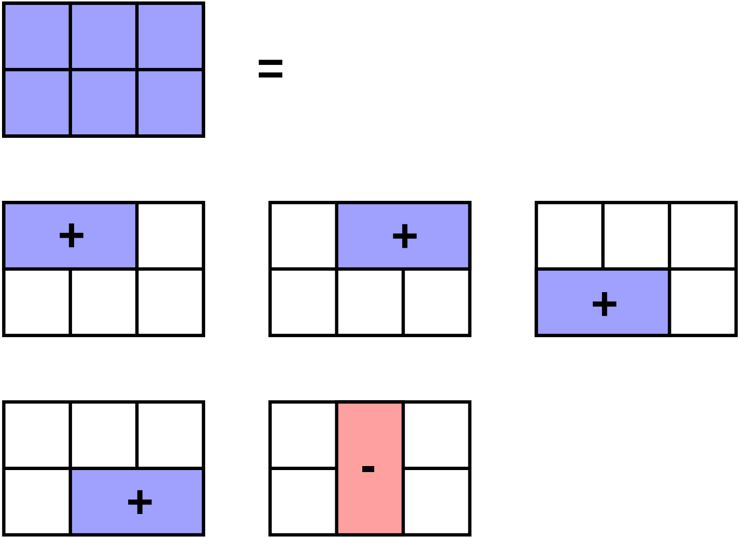
Another tiling problem

- Q: For which m, n can an $m \times n$ rectangle be tiled with copies of these tiles (rotations and reflections allowed)?



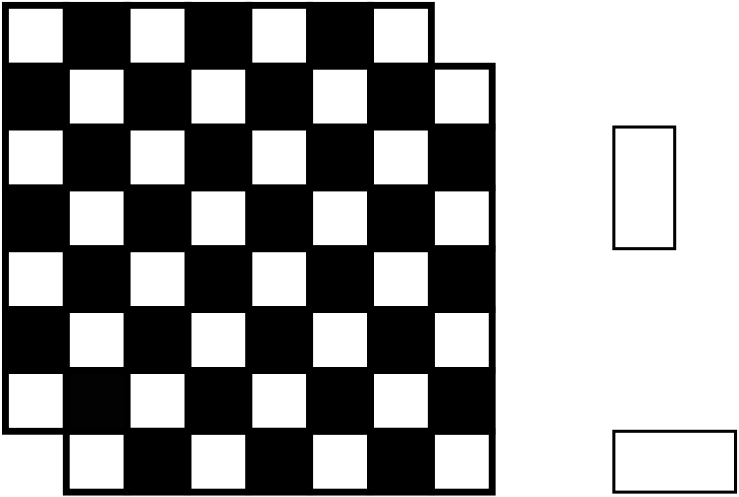
- If mn is not a multiple of 3, it seems impossible.
- Can we prove that it's impossible? Maybe by using a coloring argument?
- I claim that a coloring argument can't work.

Signed tilings



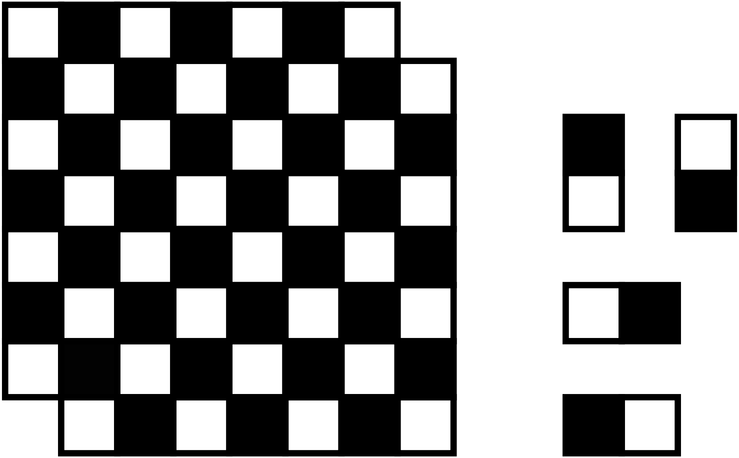
Signed tilings

- Q: Does this board have a signed tiling by dominoes?



Signed tilings

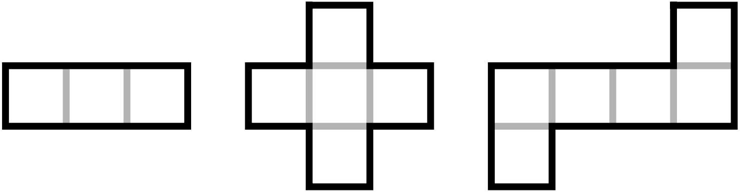
- Q: Does this board have a signed tiling by dominoes?



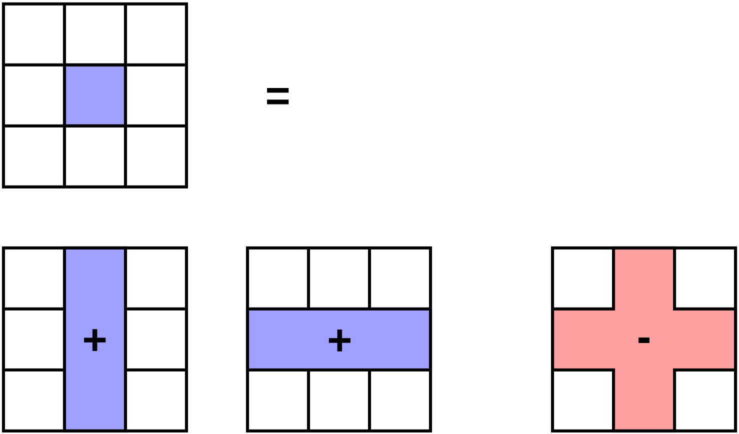
- A: No, signed tilings still cover the same number of black and white dominoes.

Signed tilings

- Q: For which m, n does an $m \times n$ rectangle have a signed tiling using copies of these tiles?



Signed tilings



- A 1×1 rectangle has a signed tiling. Then every rectangle has a signed tiling.

Signed tilings - conclusion

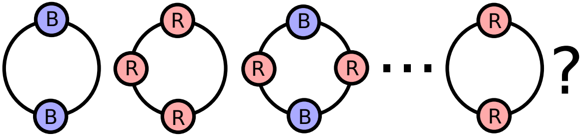
- If a coloring argument proves that a tiling can't exist, then it also proves that a signed tiling can't exist.

- So, if there is a signed tiling, coloring arguments can't help us!

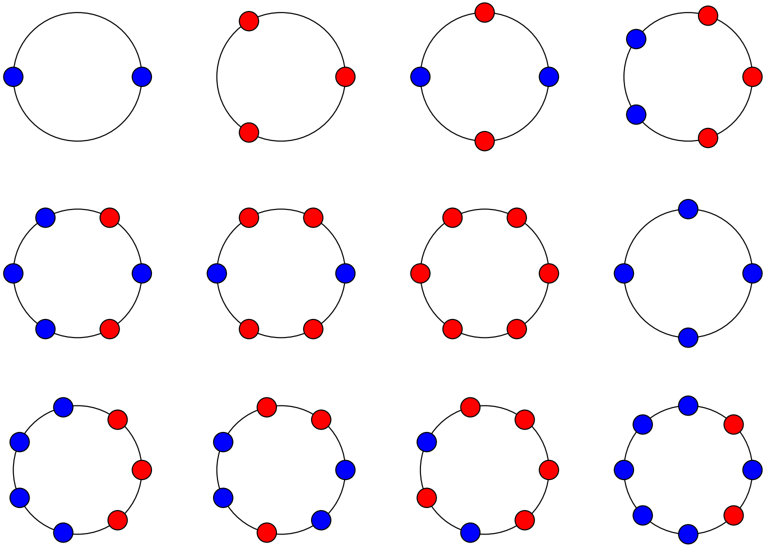
Not (obviously) a tiling problem

(Tournament of the Towns, 1980)

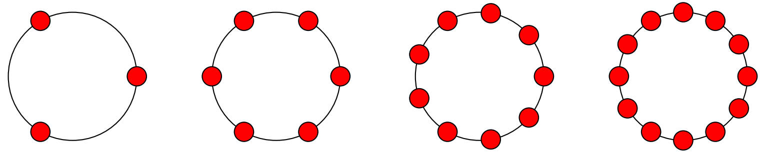
- Suppose we have a circle with red and blue beads.
- We are allowed to add a red bead to the circle and change the color of both of its neighbors, or remove a red bead from the circle and change the color of both of its neighbors.
- If we start with 2 blue beads and no red beads, is it possible to obtain a configuration with 2 red beads and no blue beads?



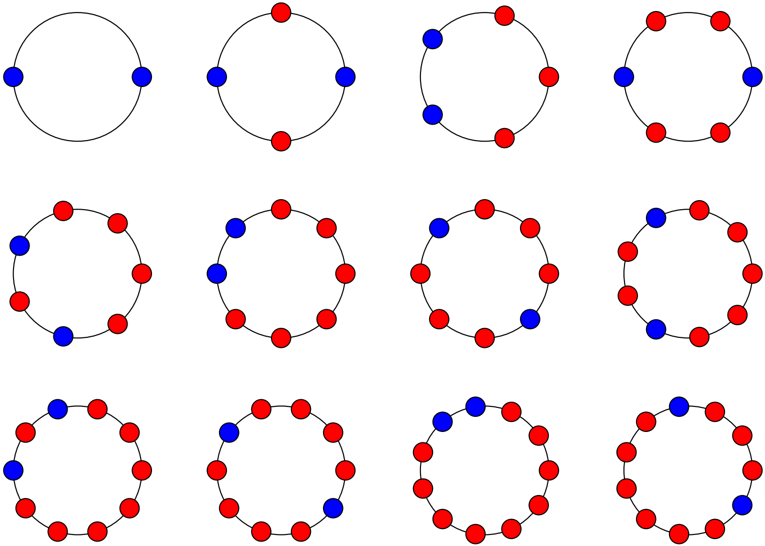
Trying all possible moves



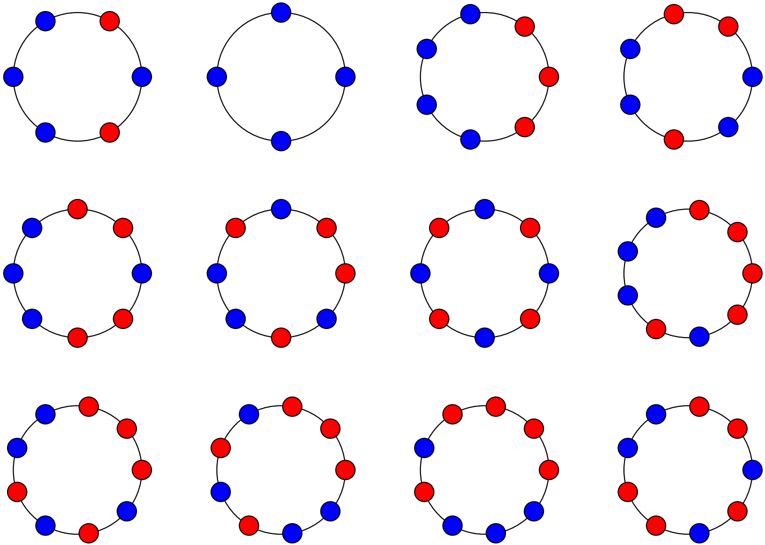
Looking for patterns



Looking for patterns

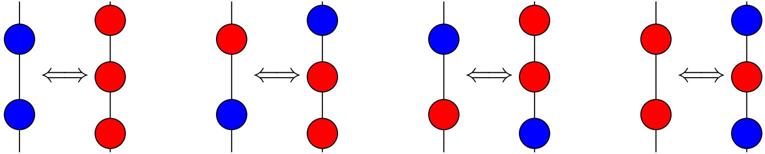


Looking for patterns



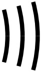
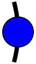
Conclusion

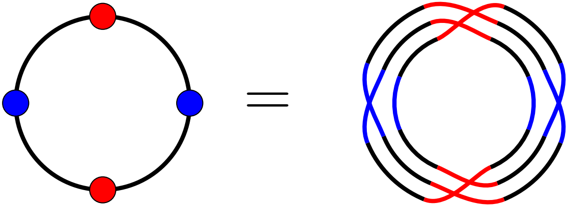
- We observed two *invariants*:
  - Number of blues is even.
  - Alternating sum of reds is divisible by 3.
- We can verify that all moves preserve the invariant:



- Can we ever get 2 reds and no blues?
  - **No**, because the alternating sum of reds would not be divisible by 3.

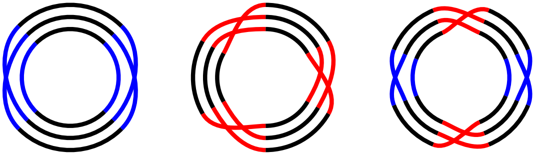
Another interpretation

- Replace  with ,  with ,  with .

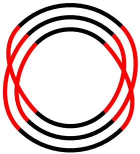


Another interpretation

- Some reachable configurations:



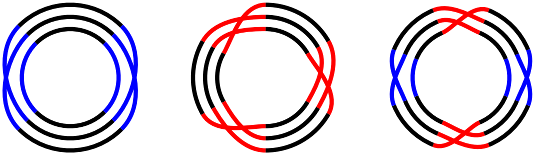
- An unreachable configuration (2 red beads):



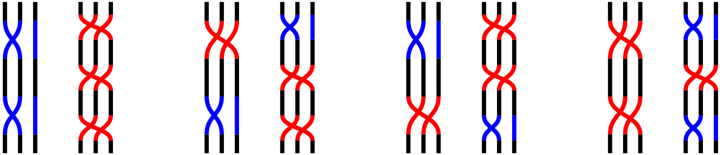
- Do you notice a difference?
  - The top three drawings have 3 strands. The bottom drawing has only 1 strand.

Another interpretation

- We found another way of describing the invariant:
  - The drawing has three separate strands.
  - In other words, if you go once around the circle, you always end up back where you started.



- Again, we can check that all moves preserve how the strands are rearranged:



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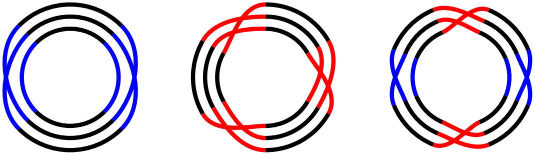
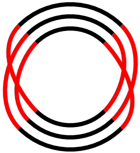
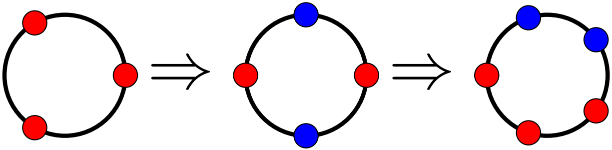


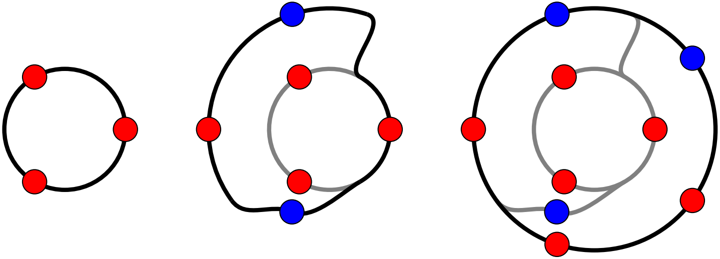
- The necklace with 2 red beads and no blue beads can never be reached, because it has only one strand.

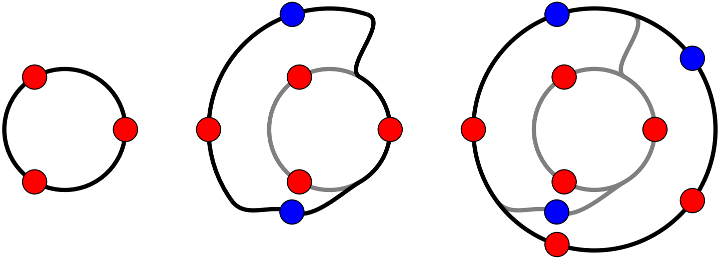


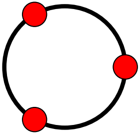
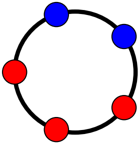
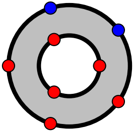
A tiling problem?

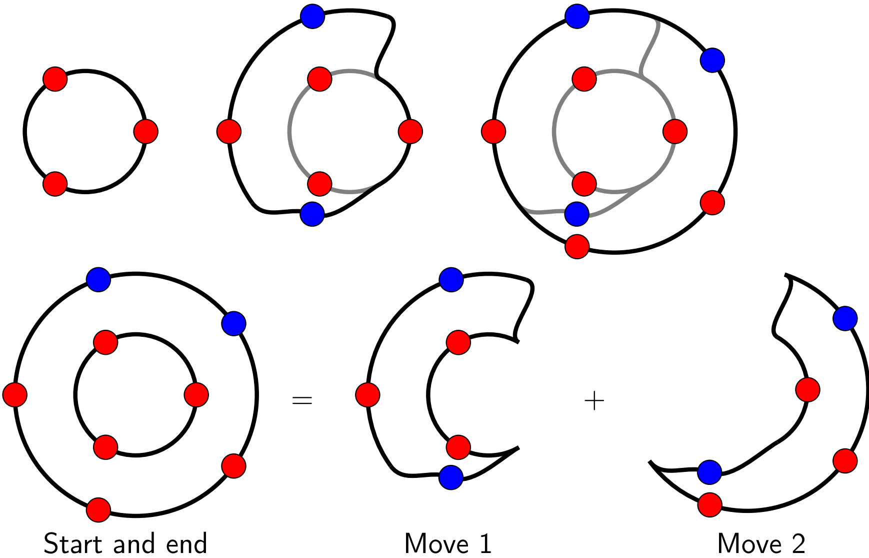
- Some allowed moves:



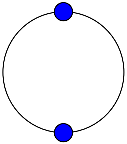
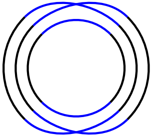


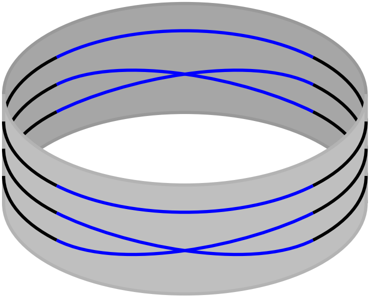


- We can think of a sequence of moves from  to  as a tiling of , with one tile per move.


=
+
Start and end
Move 1
Move 2

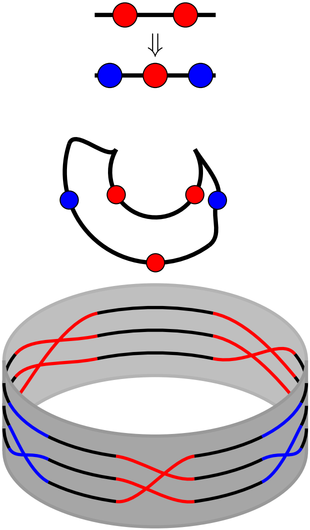
Necklaces and strands, again

- Before, we associated  with  .
- Now, we'll draw the diagram slightly differently, on a cylinder:



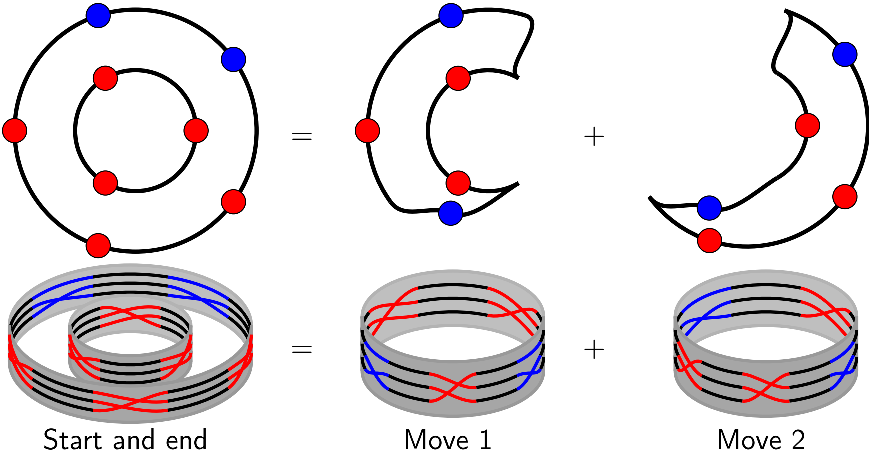
Moves and strands

- We will make a tile for each move.



- These tiles have 3 separate strands.

Smashing tiles together

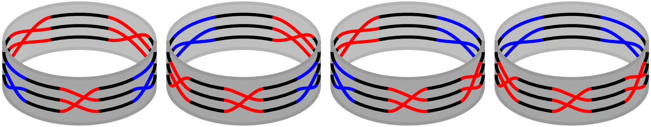


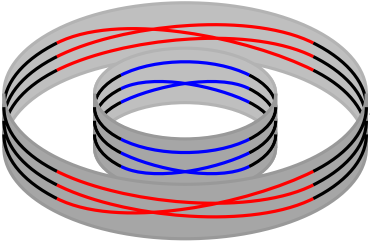
- Stretching tiles is allowed. Decorations must match.

(Actually) a tiling problem

(Tiling version of Tournament of the Towns problem)

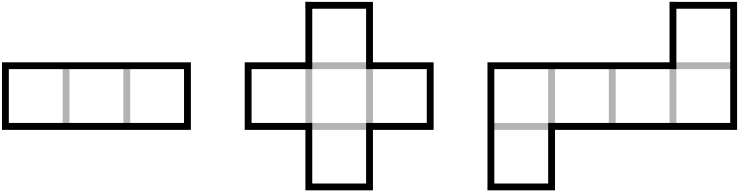
- Q: Can we combine any number of



(+ mirror images) to make  ?

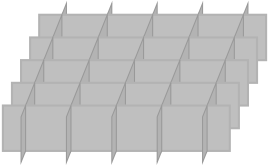
  - **No.** If the inner boundary and the tiles have three separate strands, then the outer boundary must have three separate strands as well.

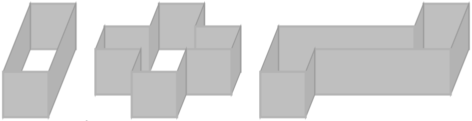
Back to the earlier problem



- It seems impossible to tile an $m \times n$ rectangle with copies of these tiles (including rotations and reflections) if mn is not a multiple of 3. Can we prove that it's impossible?

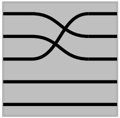
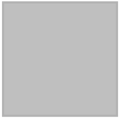
A strategy

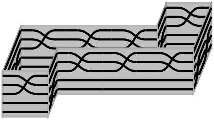
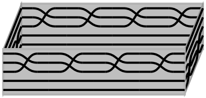


- Suppose we draw designs on the walls above, so that:
  - Going around  (or rotations or reflections) preserves the order of the strands.
  - Going around an $m \times n$ rectangle changes the order of the strands.
- Then this rectangle cannot be tiled.
- How to choose the designs?
  - Trial and error sometimes works. A computer can try lots of possibilities.

A solution

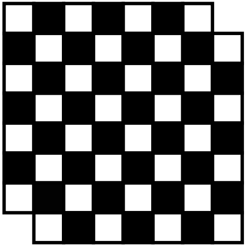
- The following happens to work:

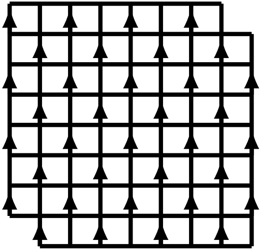
  Draw  on each ,  on each .

- Example tile: , rectangle: .
- Going around any tile preserves the order of the strands.
- Going around an $m \times n$ rectangle preserves the order of the strands if and only if mn is a multiple of 3.
- Hence $m \times n$ rectangles can only be tiled if mn is divisible by 3. This is what we wanted to prove!

Relation to coloring argument

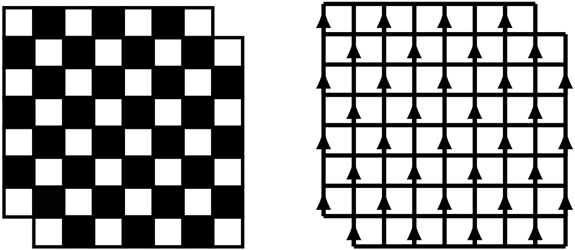
- The permutation method is actually a generalization of the coloring method.
- Here is another way of thinking about the domino tiling problem:





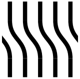
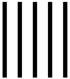
- Black squares have 1 counterclockwise arrow around their boundary, and white squares have 1 clockwise arrow around their boundary.
- Any domino has the same number of clockwise and counterclockwise arrows around its boundary.

Relation to coloring argument



- Black squares have 1 counterclockwise arrow around their boundary, and white squares have 1 clockwise arrow around their boundary.
- Any domino has the same number of clockwise and counterclockwise arrows around its boundary.
- So any tileable region has an equal number of clockwise and counterclockwise arrows around its boundary.
- The chessboard with 2 corners missing has 5 clockwise arrows and 3 counterclockwise arrows around its boundary.

The domino tiling problem can be solved with permutations.

- Use infinitely many strands.
- Edges marked with an arrow: ...  ...
- All other edges: ...  ...
- When going around a closed loop, strands get shifted by # of counterclockwise arrows − # of clockwise arrows.

Further reading

- J. Conway and J. Lagarias, "Tiling with Polyominoes and Combinatorial Group Theory"
- D. Fuchs and S. Tabachnikov, "Impossible Tilings", in *Mathematical Omnibus: Thirty Lectures on Classic Mathematics*
- J. Propp, "A Pedestrian Approach to a Method of Conway, or, A Tale of Two Cities"
- W. Thurston, "Conway's tiling groups"